AVERAGED MODELING OF CONVERTERS OPERATING IN CONTINUOUS AND DISCONTINUOUS CONDUCTION MOD – REDUCED ORDER MODEL

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Abstract: The difference between the converters dc-dc, which operate in continuos and discontinuous modes is that the latter has three functioning stages for a cycle of switching – switch ON, switch OFF and stage of zero current. Moreover, there are new restrictions to the functioning of the circuit, namely the momentary value of the current through the inductance where the energy accumulated begins to grow above zero at the beginning of the commutation cycle and reaches zero before the commutation cycle ends.

I. THE STATE – SPACE AVERAGING OF THE BASIC MODEL

At first we have to describe from the point of view of time a switching converter function in DCM, represented in Figure 1

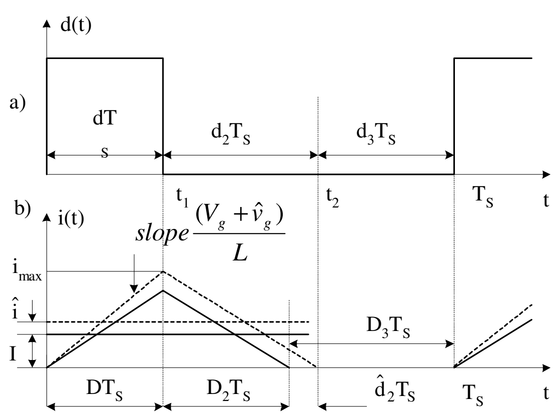

Figure 1. Description of the time intervals and of the magnitude of the perturbations:
a) switching states of the transistor.
b) wave form of the current through the inductance.

As we can notice from the Figure 1, the switching interval OFF $[t_1, T_S]$ is split in two intervals d_2T_S and d_3T_S (or D_2-T_S and D_3T_S).

The duration of the first interval is dictated by the command input signal of the switch in ON state and can be controlled exactly by the command circuit. And the second interval, d_2T_S (or D_2T_S), called the discharging interval cannot be known because, generally, it depends on the circuit parameters.

For each of the three intervals there are equivalent circuits of the converter which can be described by the state equations.

$$\begin{aligned} \dot{x} &= A_1 x + b_1 v_g \quad for \quad d_1 T_S, (0 \le t \le t_1) \\ \dot{x} &= A_2 x + b_2 v_g \quad for \quad d_2 T_S, (t_1 \le t \le t_2) \\ \dot{x} &= A_3 x + b_3 v_g \quad for \quad d_3 T_S, (t_2 \le t \le T_S) \end{aligned} \tag{1}$$

While for the converter in (Continuos Conduction Mode) CCM, these equations would suffice to completely describe its functioning, in the (Discontinuous Conduction Mode) DCM, the functioning is not completely described because the momentary current through the inductance has some constraints during the switching cycle.

$$\begin{aligned} i(0) &= i[(d_1 + d_2)T_S] \\ i(0) &= 0 \quad for \quad t \in [t_2, T_S] \end{aligned} \tag{2}$$

Thus, the equations (2), offer a complete description of the functioning of the switching converter.

$$\begin{aligned} \dot{x} &= (d_1 A_1 + d_2 A_2 + d_3 A_3) x \\ &+ (d_1 b_1 + d_2 b_2 + d_3 b_3) v_g \end{aligned} \tag{3}$$

The constraints in the equations (2) are caused by the fact that the secondary duty factor d_2 depends on the state of the circuit and on the control variables. For a better averaged model of functioning in DCM we have to eliminate the variable d_2 by expressing it through the averaged variables of voltage and current.

II. THE DYNAMIC MODEL OF THE CONVERTER IN DCM

Starting from the averaged state equations (3), we pass to the small-signal system and we obtain the following equations:

$$\dot{\hat{x}} = A\hat{x} + b\hat{v}_g + f\hat{d} + g\hat{d}_2 \tag{4}$$

where:

$$A = (A_1 - A_3)D + (A_2 - A_3)D_2 + A_3$$
$$b = (b_1 - b_3)D + (b_2 - b_3)D_2 + b_3$$
$$f = (A_1 - A_3)X + (b_1 - b_3)V_g$$
$$g = (A_2 - A_3)X + (b_2 - b_3)V_g$$

Equation (4) is a function depending on $\hat{d}_2$:

$$\dot{\hat{x}} = f_1\left(\hat{x}, \hat{v}_g, \hat{d}, \hat{d}_2\right) \tag{5}$$

Using the current conditions through the induction we can find a relation between $\hat{d}$ and $\hat{d}_2$:

$$D_2 = f_2(D, V_g, V_0) \tag{6}$$

when we introduce the small-signal perturbations in (6), we obtain:

$$d_2 = f_3\left(\hat{d}, \hat{v}_g, \hat{v}_0\right) \tag{7}$$

after substituting the equation (7) in (5), we obtain:

$$\dot{\hat{x}} = f_4\left(\hat{x}, \hat{v}_g, \hat{d},\right) \tag{8}$$

If the last operation is done correctly, the stored energy during the switching cycle will be zero, because the change in the inductor current from the beginning to the end of the cycle is zero.
The output equations will be as follows:

$$\hat{v}_0 = f_5\left(\hat{i}, \hat{d}, \hat{v}_g, \hat{v}_0\right) \tag{9}$$

The obtained equation is not yet under the form we are looking for because the variable $\hat{i}$ is present. The inductor current can be eliminated from the output equation as follows:

$$i = \frac{i_{max}}{2} = f_6\left(d, v_g, v_0\right) \tag{10}$$

which, when we introduce the perturbation, becomes:

$$\hat{i} = f_7\left(\hat{d}, \hat{v}_g, \hat{v}_0\right) \tag{11}$$

after we substitute the equation (11) in to (9), we obtain the output equation:

$$\hat{v}_0 = f^*\left(\hat{d}, \hat{v}_g, \hat{v}_0\right) \tag{12}$$

III. AN EXAMPLE OF MODELING OF THE BUCK-BOOST CONVERTER IN DCM

The following figure presents the buck-boost converters functioning in DCM, represented as a three state switch.

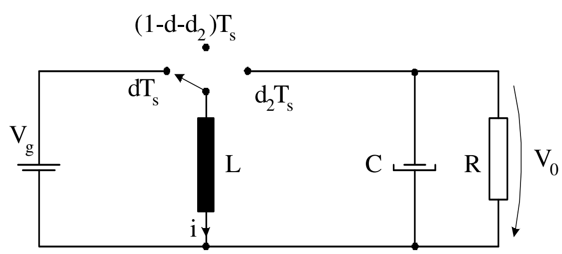

Figure 2. Ideal buck-boost converter

Here are the equations for the three functioning states:
the interval dTs:

$$\frac{di}{dt} = \frac{v_g}{L}, \quad \frac{dv_0}{dt} = -\frac{v_0}{RC} \tag{13}$$

the interval d_2T_S:

$$\frac{di}{dt} = \frac{v_0}{L}, \quad \frac{dv_0}{dt} = -\frac{i}{C} - \frac{v_0}{RC} \tag{14}$$

the interval $(1-d-d_2)T_S$:

$$L\frac{di}{dt} = 0, \quad \frac{dv_0}{dt} = -\frac{v_0}{RC} \tag{15}$$

The matrices of the state equations for the three functioning states:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{pmatrix};\; A_2 = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{pmatrix}$$

$$b_1 = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix};\; b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\; b_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \frac{D_2}{L} \\ -\frac{D_2}{C} & -\frac{1}{RC} \end{pmatrix};\; b = \begin{pmatrix} \frac{D}{L} \\ 0 \end{pmatrix} \qquad (16)$$

The stationary solution is:

$$X = -A^{-1} b V_g == -\frac{LC}{D_2^2} \begin{pmatrix} -\frac{1}{RC} & -\frac{D_2}{L} \\ -\frac{D_2}{C} & 0 \end{pmatrix} \begin{pmatrix} \frac{D}{L} \\ 0 \end{pmatrix} V_g \qquad (17)$$

$$X = \begin{pmatrix} \frac{DV_g}{RD_2^2} \\ -\frac{DV_g}{D_2} \end{pmatrix} \qquad (18)$$

$$V_0 = -\frac{D}{D_2} V_g;\quad I = -\frac{V_0}{RD_2} \qquad (19)$$

The small-signal model of the converter is:

$$\dot{\hat{x}} = A\hat{x} + b\hat{v}_g + f\hat{d} + g\hat{d}_2 \qquad (20)$$

$$f = (A_1 - A_3)X + (b_1 - b_3)V_g = \begin{pmatrix} \frac{V_g}{L} \\ 0 \end{pmatrix} \qquad (21)$$

$$g = (A_2 - A_3)X + (b_2 - b_3)V_g = \begin{pmatrix} -\frac{DV_g}{D_2 L} \\ -\frac{DV_g}{RCD_2^2} \end{pmatrix} \qquad (22)$$

The detailed state equations:

$$\dot{\hat{x}} = \begin{pmatrix} 0 & \frac{D_2}{L} \\ -\frac{D_2}{C} & -\frac{1}{RC} \end{pmatrix} \hat{x} + \begin{pmatrix} \frac{D}{L} \\ 0 \end{pmatrix} \hat{v}_g + \begin{pmatrix} \frac{V_g}{L} \\ 0 \end{pmatrix} \hat{d} + \begin{pmatrix} -\frac{DV_g}{D_2 L} \\ -\frac{DV_g}{RCD_2^2} \end{pmatrix} \hat{d}_2 \qquad (23)$$

The variable $\hat{d}_2$ in (23) can be eliminated for the following reasons:

$$i_{\max} = \frac{v_g dT}{L} = -\frac{v_0 d_2 T_S}{L} \qquad (24)$$

$$\hat{d}_2 = -\frac{V_g}{V_0}\hat{d} - \frac{D}{V_0} v_g - \frac{D_2}{V_0}\hat{v}_0 \qquad (25)$$

After eliminating $\hat{d}_2$, the state equations (23) are reduced to:

$$\dot{\hat{x}} = \begin{pmatrix} 0 & 0 \\ -\frac{D_2}{C} & -\frac{1}{RC} \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ -\frac{D}{D_2 RC} \end{pmatrix} \hat{v}_g + \begin{pmatrix} 0 \\ -\frac{V_g}{D_2 RC} \end{pmatrix} \hat{d} \qquad (26)$$

As we have expected, the first row of the matrices is zero. It means that the changing speed of the current through the inductor (the derived) is zero, and this can no longer be considered as a state variable of the system.

$$\hat{i} = 0 \qquad (27)$$

$$\hat{v}_0 = -\frac{V_g}{D_2 RC}\hat{d} - \frac{D}{D_2 RC} v_g - \frac{2}{RC}\hat{v}_0 - \frac{D_2}{C}\hat{i} \qquad (28)$$

The variable $\hat{i}$, can be eliminated from the equation (28) because of the following reason:

$$I = \frac{DV_g}{RK};\quad K = \frac{2L}{RT_S};\quad \hat{i} = \frac{V_g}{KR}\hat{d} + \frac{D}{KR}\hat{v}_g \qquad (29)$$

The result for $\hat{i}$ from the equation (29) is substituted in the equation (28), and thus, we obtain the final output equation of the converter.

$$\hat{v}_0 = -\frac{2V_g}{D_2RC}\hat{d} - \frac{2D}{D_2RC}v_g + \frac{2}{RC}\hat{v}_0 \qquad (30)$$

From the output equation (30) we obtain the transfer function in the frequency domain of the buck-boost converter in DCM.

The command transfer function:

$$\frac{\hat{v}_0(s)}{\hat{d}(s)} = -\frac{V_g}{\sqrt{K}}\frac{1}{1+\frac{s}{\omega_p}};$$
$$K = \frac{2L}{RT_S};\ \omega_p = \frac{2}{RC} \qquad (31)$$

The input transfer function:

$$\frac{\hat{v}_0(s)}{\hat{v}_g(s)} = -\frac{M}{1+\frac{s}{\omega_p}};\ M = \frac{V_0}{V_g} \qquad (32)$$

We have to notice that, because the elimination of the current through the inductor from the state variable, the order of the converter is reduced by one.

IV. THE ANALYSIS OF THE CONVERTORS IN DCM USING A SIMULATIONS MODEL

Equations, tables and figures will be included in the text in the order of reference.

In DCM the converter will have three functioning states, and the state equations will be obtained from the averaging of the state equations describing each state (equations 1).

Thus we obtain the following state equations.

$$\dot{x} = (d_1A_1 + d_2A_2 + d_3A_3)x + (d_1b_1 + d_2b_2 + d_3b_3)v_g$$
$$+ (d_1e_1 + d_2e_2 + d_3e_3)v_T + (d_1n_1 + d_2n_2 + d_3n_3)v_D$$
$$v_0 = (d_1q_1 + d_2q_2 + d_3q_3)x \qquad (33)$$

The duty factor d is the command-input signal, and d_2 and d_3 will be obtained in real time using d and the variables typical of the circuit.

The functioning diagram of the model is described in the following figure.

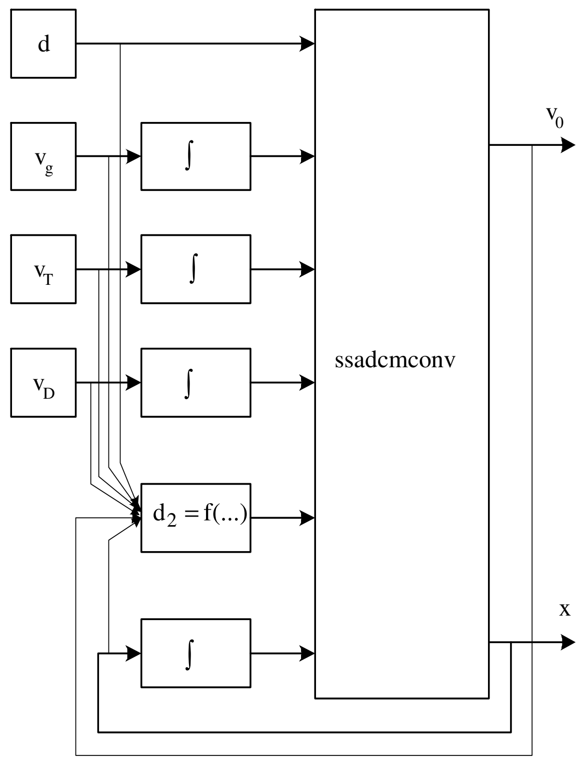

Figure 3. Functioning diagram of the converter in DCM

The ssadcmconv function is an external function which calculates the averaged matrices of the state and output equations and which give the voltage output and the state vector at the output. The listing of the function is presented in what follows.

```
function  y=ssadcmcconv(u)

d = u (1);
ivg = u (2);
ivT = u (3);
ivD = u (4);
d2 = u (5);
d3 = 1- d - d2;
ix =[ u (6); u(7)];

global A1  A2  A3  b1  b2  b3  q1  q2  q3  e1  e2  e3  n1
n2  n3

A  =  A1*d  +  A2*d2  +  A3*d3;

b   =  b1*d  +   b2*d2  +  b3*d3;

q   =   q1*d  +  q2*d2  +  q3*d3;

e   =   e1*d  +  e2*d2  +  e3*d3;

n   =   n1*d  +  n2*d2  +  n3*d3;
```

*x = A*ix + b*ivg + e*ivT + n*ivD;*

*v0 = q*x;*

y = [0 0 0];
y (1) = x (1);

y (2) = x (2);
y (3) = v0;

From the functioning diagram presented in Figure 3 , we pass easily to the Simulink model like in Figure 4.

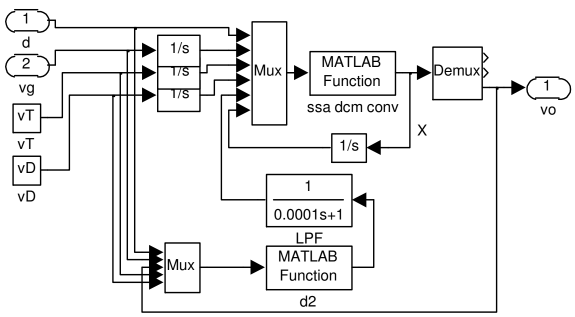

Figure 4. Simulink model for a cc-cc converter in DCM

Using (24) which expresses the maximum value of the current through the inductor we can obtain the relation for d_2 on the basis of d and of the input and output voltages.

$$i_{\max} = \frac{v_g dT_s}{l} = -\frac{v_0 d_2 T_s}{L} \Rightarrow d_2 = d\frac{V_g}{|V_0|} \qquad (34)$$

If we take into account the voltage drops on the switch in ON and OFF states, we have to substract from V_g the voltage drop in ON state (v_T), and to add to V_O the voltage drop in OFF state (v_D). The relation for d_2 becomes:

$$d_2 = d\frac{V_g - V_T}{|V_0 + V_D|} \text{ when } d_2 < 1 - d \text{ (in DCM)} \qquad (35)$$

Finally, we obtain the relation which can be used in Simulink:

$$d_2 = \min(1-d,\ d(V_g - V_T)/abs(V_0 + V_D)) \qquad (36)$$

The Matlab function from the model in Figure 4 for obtaining d_2 is:

min([1-u(1),u(1)*(u(2)-u(4))/max([abs(u(3)+eps+u(5) eps])])

(37)

Figure 5 presents the circuit of the boost converter which is used in the simulation (MATLAB) and which contains the real elements of the circuit.
V_T and V_D are the voltage drops on the switch in states ON and OFF.
L and C are the reactive elements in the circuit.
V_g and R_g represent the power source and its parasite resistance, respectively.

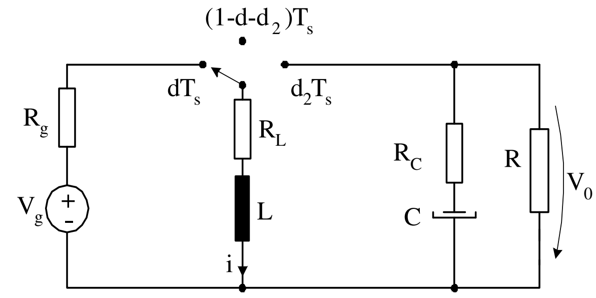

Figure 5. The real circuit of the buck-boost converter

Having the following circuit elements: L=10mH, C=50uF, Vg=50V, R_L=0.2 ohm, R_C=0.3ohm, R_g=0.3ohm, V_T=0.6V si V_D=0.4V, Figure 6 presents the response in time of time of the output voltage v_0 and of d_2, for duty factor d = 0,4 and for a load resistance R = 27 ohm.

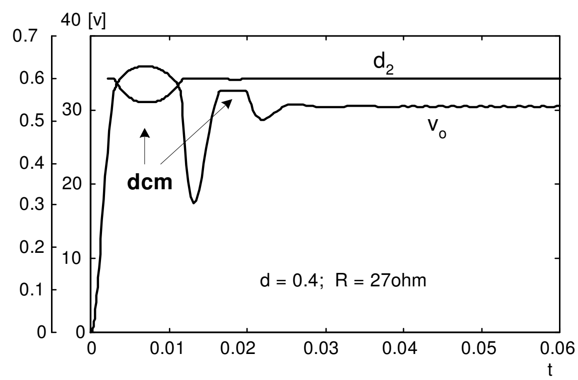

Figure 6. The response in time of the v_0 voltage and of d_2

The figure also presents the trajectory of d_2 and v_0 in the transitory regime of the converter until the circuit establishes the stationary functioning regime. In the transitory regime the converter passes several times from CCM regime in DCM regime and vice versa, depending on the excitation level at the input.
In Figure 6, we notice that, when d = 0,6 the converter works in CCM regime, but, because of the increase of the voltage v_0, d_2 diminishes and the converter enters in DCM regime.

The analysis of the converter in DCM that we presented is an averaged reduced order model, because the derived of the current through the inductor is zero. Thus we have a constant state variable ($di_L/dt=0$) which leads us to a degenerate model where the dynamics of the current through the inductor disappears. Thus, the averaged reduced order model is independent of the dynamic of the current through the inductor which determines important limitations.

The modeling described in the paper contributes to the efficiency of the simulation because both functioning regimes (DCM and CCM) have been taken into account.

V. CONCLUSIONS

In this article we described the space-state averaged modeling of the converters operating in DCM.

To the averaged state equations we add a supplementary restriction when the switch is OFF. This period is obtained from the conservation law volt-second. Thus the average current through to the inductor is excluded from the state variables as we do not take into account the variation of the current through the inductor.

Because of these approximations the order of the circuit is reduced by one which leads to important limitations in the transfer characteristic at high frequencies.

Nevertheless at low frequencies the model could be quite exact and because of its simplicity, it can be used very efficiently.

We have to say that, because of the implementation method of the model in MATLAB it can function very well in DCM as well as in CCM.

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